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ANALYTICAL AND NUMERICAL INVESTIGATION OF TRANSONIC AND SUPERSO--ETC(U)
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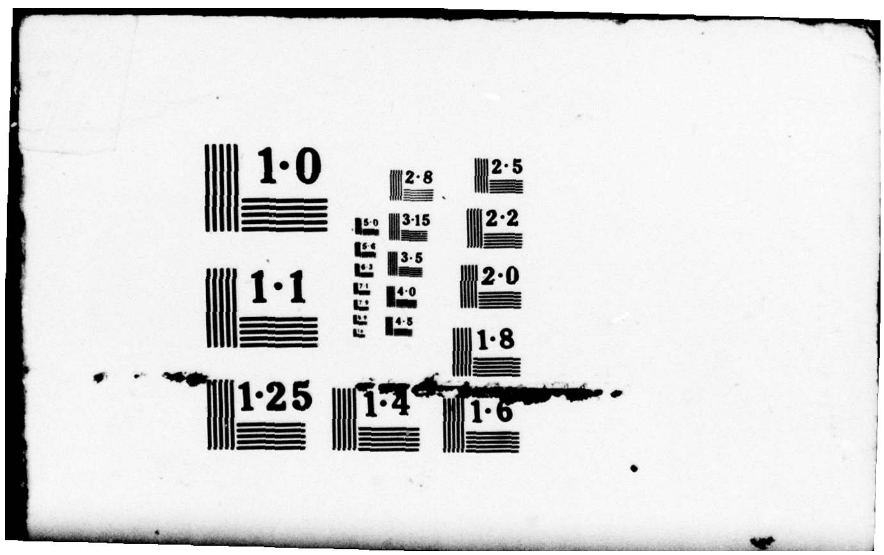
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Introduction

In the following sections we discuss the scientific accomplishments made during the course of the 19 1/2 months of this contract. All the work was in the area of gasdynamics, and our main area of research has been cases of flows containing shockwaves. For the most part we have taken an approach which straddled the borderline of analytical and numerical methods. This is best exemplified by the work discussed: in Sec. 1 in which classical boundary layer methods are used to develop variable mesh numerical schemes; and in Secs. 2 and 3 in which the theory of characteristics, Riemann invariants, and shock expansion theory are all used to obtain a robust method of numerical integration which is orders of magnitude faster than any other known scheme.

During the contract period the four following manuscripts have been prepared:

1. "A Variable Mesh Finite Difference Method for Solving a Class of Parabolic Differential Equations."
2. "An Approximate Solution in Gasdynamics."
3. "On the Numerical Integration of the Gasdynamic Equations."
4. "On the Equations Governing a Trailing Wake."

Of these the first has been published in the SIAM Journal on Numerical Analysis, and the other three have been submitted for publication. Two additional manuscripts are in preparation:

1. "Two-Dimensional Supersonic Flow: I. Analytical Approximation."
2. "Two-Dimensional Supersonic Flow: II. Exact Numerical Integration."

In addition to the work covered by these manuscripts, several other projects were undertaken but were not felt to warrant a publication. These are also discussed below.

1. Numerical Integration of Equations of the Navier-Stokes Type

A number of situations in fluid mechanics lead to flows containing "boundary layers" where flow variables experience rapid variation. In the case of shock waves a priori location of this region is not known (this is also true of the wake roll-up region behind a moving body). The numerical integration of the fluid equations in such situations present a number of difficulties. In any real situation, where Reynolds numbers are large, transition layers are small and a numerical scheme based on a uniform mesh is unfeasible. To deal with this situation we have investigated a variable mesh scheme for the equation

$$\frac{\partial^2 u}{\partial x^2} = F(x, t, u, u_x, u_t) \quad . \tag{1}$$

From the mathematical viewpoint this represents a parabolic partial differential equation and it is known that such equations, even for smooth data, give rise to regions of rapid variation for which a priori error bounds fail.

A particular example of equation (1) is

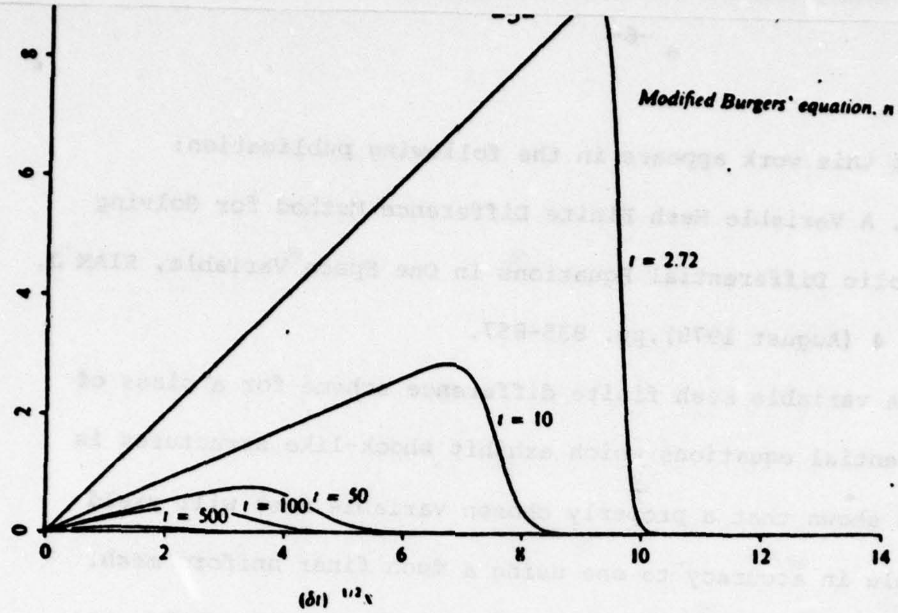
$$\frac{\partial u}{\partial t} + kt^{-n/2} u \frac{\partial u}{\partial x} = \frac{\delta}{2} \frac{\partial^2 u}{\partial x^2} \tag{2}$$

which for n = 1 is the equation of Burgers [1]. More recently Eqn. (2), which we call the modified Burgers equation, has been derived from the Navier-Stokes equations for problems involving explosions and sonic booms [2], [3].

We have developed a method for dealing with problems of this type,

which is uniformly second order accurate. A variable mesh (in x) is generated based on boundary layer methods. The neighborhood of the shock contains a zone of high density grid points, and a low density grid is used in regions of slow variation. The grid system is rezoned during the course of the numerical integration, i.e., grid generation is an integral part of the numerical code. Further reduction of the numerical problem is obtained from use of approximate shock trajectories, which can be derived from asymptotic theory.

The results of this investigation extend by three orders of magnitude a similar calculation (using a uniform mesh) found in [4]. As an example if $n = 2$ and the reciprocal Reynolds number δ is 10^{-5}, a uniform mesh approach requires $\geq 10^5$ mesh points. For the method discussed here 600 points are required at the initial instant. With the course of time this decreases steadily, e.g., at $t = 2000$ only 300 points are required. The following figures give in graphical form the evolution of an initial discontinuity for the three Reynolds numbers, $R = \delta^{-1} = 10^3, 10^4, 10^5$.


Modified Burgers' equation, n = 2, δ = 10⁻³.
t = 2.72
t = 10
t = 50
t = 100
t = 500
(δt)^{-1/2}u
(δt)^{1/2}x
0
2
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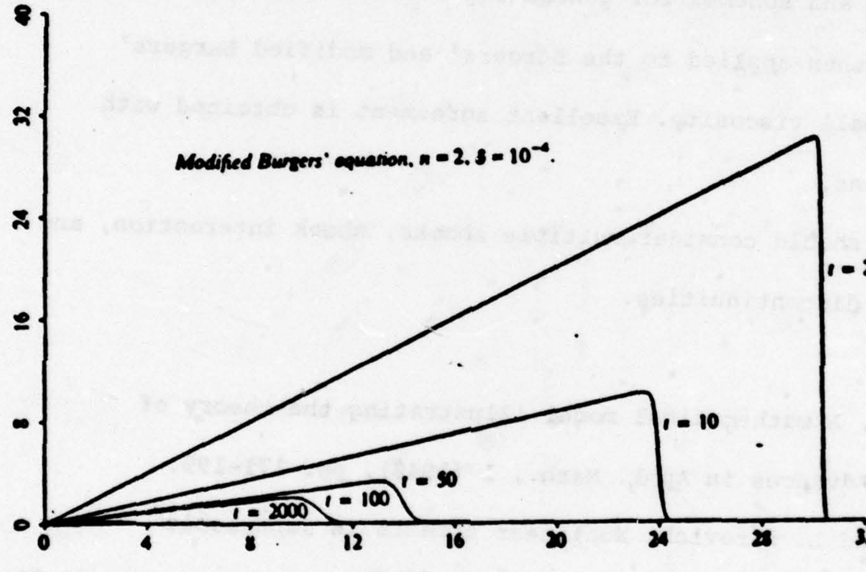
Modified Burgers' equation, n = 2, δ = 10⁻⁴.
t = 2.72
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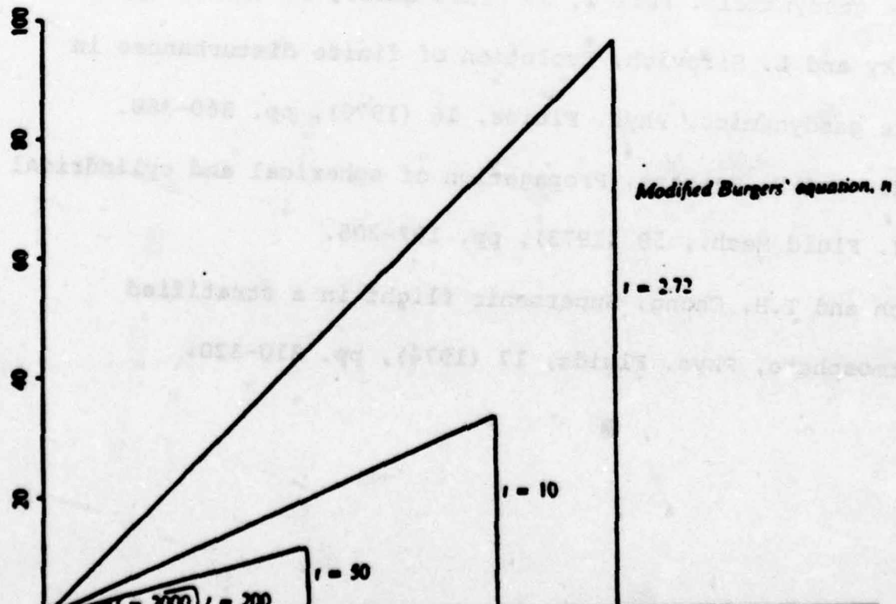
Modified Burgers' equation, n = 2, δ = 10⁻⁵.
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0
20
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100
120
140

A report of this work appears in the following publication:

T.H. Chong, A Variable Mesh Finite Difference Method for Solving a Class of Parabolic Differential Equations in One Space Variable, SIAM J. Numer. Anal. 15, 4 (August 1978),pp. 835-857.

Abstract. A variable mesh finite difference scheme for a class of parabolic differential equations which exhibit shock-like structures is developed. It is shown that a properly chosen variable mesh will yield results comparable in accuracy to one using a much finer uniform mesh. Computable criteria and schemes for generating such variable meshes are given. A scheme is then applied to the Burgers' and modified Burgers' equations with a small viscosity. Excellent agreement is obtained with known exact solutions.

Further work should consider multiple shocks, shock interaction, and higher dimensional discontinuities.

References

[1] J.M. Burgers, A mathematical model illustrating the theory of turbulence, Advances in Appl. Mech., 1 (1948), pp. 171-199.

[2] T.H. Chong and L. Sirovich, Nonlinear effects in supersonic dissipative gasdynamcis. Part 2, J. Fluid Mech., 58 (1973), pp. 53-63.

[3] L. Halabisky and L. Sirovich, Evolution of finite disturbances in dissipative gasdynamics, Phys. Fluids, 16 (1973), pp. 360-368.

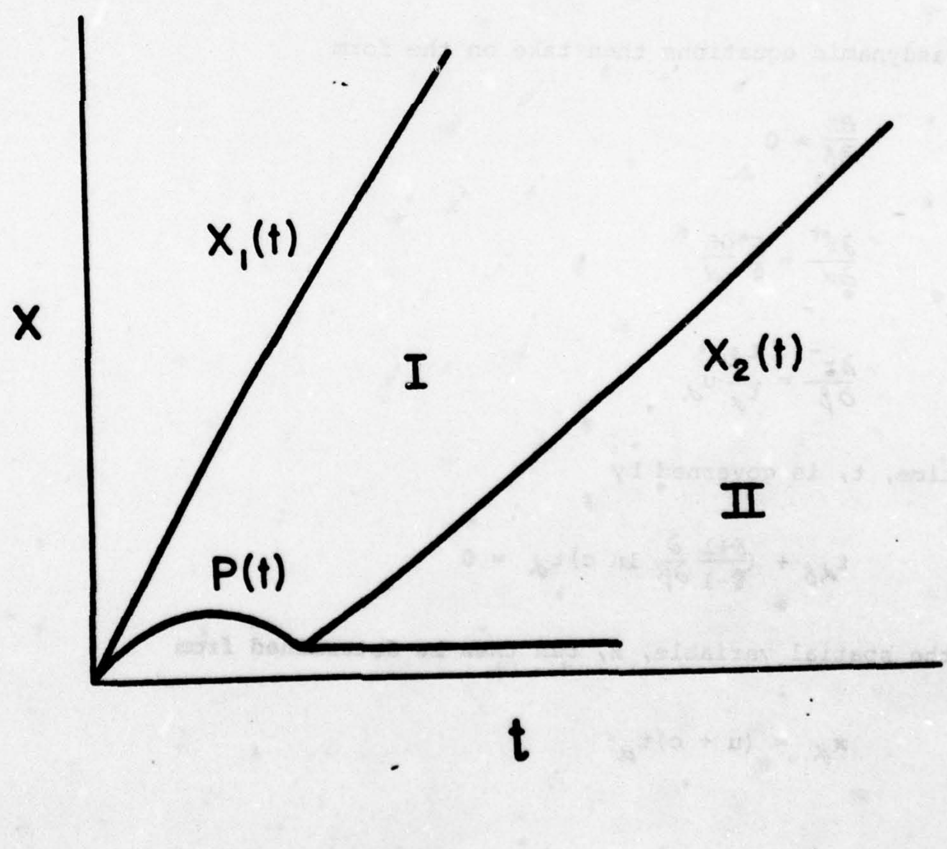
[4] P.L. Sachdev and R. Seebass, Propagation of spherical and cylindrical N-waves, J. Fluid Mech., 58 (1973), pp. 192-205.

[5] L. Sirovich and T.H. Chong, Supersonic flight in a stratified sheared atmosphere, Phys. Fluids, 17 (1974), pp. 310-320.

2, Unsteady Gasdynamics--Analytic Approximation and Numerical Integration

We have pursued a novel method of treating gasdynamic flows in which shock waves occur. Our research has produced a new analytical approximation and a new numerical procedure for the exact integration of gasdynamic flows with shocks. As a result of the analytic approach we have been able to recast the gasdynamic equations in a form which we believe is optimal for machine integration. The result has been a highly accurate and extremely rapid numerical pro· dure. Unlike existing methods, where shock waves oscillate, and often fail to stabilize, the present method is able to fix a shock wave once and for all.

We consider the situation as depicted in the figure.



A piston is moved into a gas at rest with the trajectory P(t). Leading and trailing shocks $X_1(t)$ and $X_2(t)$ are formed. In brief the analysis is as follows:

Riemann invariants (normalized)

(1) $$r^{\pm} = u \pm \frac{2c}{\gamma - 1}$$

are introduced. ((1) is for an ideal gas, the modification of the development for non-ideal gases is relatively minor.) A new coordinate system (α, β) is introduced through

$$\frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial x} = 0$$

$$\frac{\partial \beta}{\partial t} + (u + c)\frac{\partial \beta}{\partial x} = 0$$

The gasdynamic equations then take on the form

(2) $$\frac{\partial S}{\partial \beta} = 0$$

(3) $$\frac{\partial r^{+}}{\partial \alpha} = \frac{c}{\gamma}\frac{dS}{d\alpha}$$

(4) $$\frac{\partial r^{-}}{\partial \beta} = \frac{t_{\beta}}{t_{\alpha}} u_{\alpha}$$

The time, t, is governed by

(5) $$t_{\alpha\beta} + \left(\frac{\gamma+1}{\gamma-1}\frac{\partial}{\partial \beta}\ln c\right)t_{\alpha} = 0$$

and the spatial variable, x, can then be determined from

$$x_{\alpha} = (u + c)t_{\alpha}$$

These equations are to be augmented by shock and boundary conditions, which are not repeated here.

The first step in the procedure follows from an approximation which occurs in shock expansion theory [1], [2], [3], viz.,

(6) $\bar{r} \approx \bar{r}_o(\alpha)$

With adoption of (6) we drop equation (4). Eqs. (2), (3), (5), and (6) constitute the approximate theory and can be solved in terms of quadratures and the functions $S(\alpha)$, $\bar{r}_o(\alpha)$ which are evaluated at the shock wave. The results of this approximation are remarkably accurate. Even in the neighborhood of ionizing shocks the approximate and exact solutions differ by less than one percent.

The <u>exact numerical</u> procedure uses the approximate solution as a first step in an iterative procedure. The approximate solution is substituted into the right hand side of (4) and a new value of $\bar{r}(\alpha,\beta)$ obtained. This in turn is used to solve (2), (3), (5), and (6), and the iteration repeated if necessary. According to the present version of the numerical codes, this procedure converges in less than three iterations.

As an example which is typical of the integration procedure, we consider a parabolic piston motion,

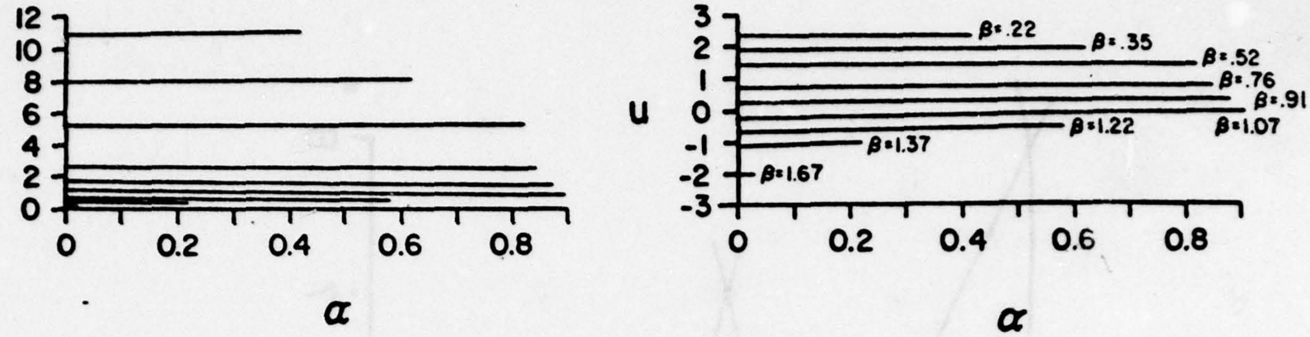
$$P(t) = M_o t(1 - t/2)$$

where M_o is the Mach number of the piston at the initial instant. At

M_o = 3 the flow is considerably away from equilibrium. For example, the temperature behind the initial shock is four times its initial value. The result of the numerical integration for M_o = 3 is shown in the following four figures.

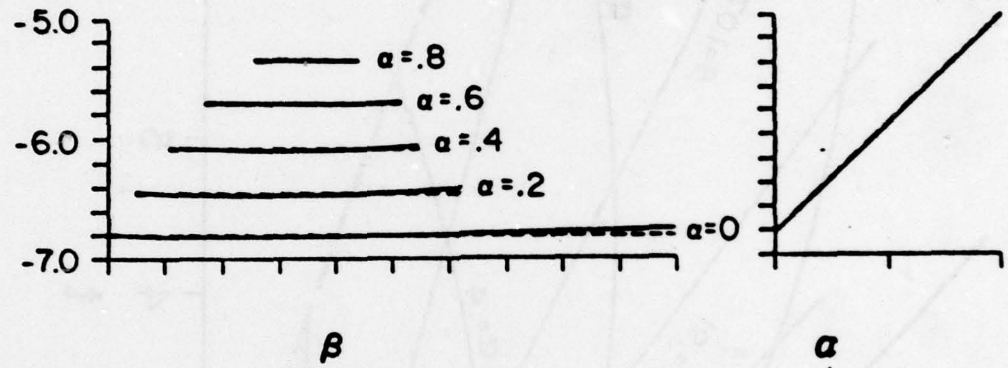
The first of these figures indicates the particle paths, α equal constant, the C^+ characteristics, β equal constant, and the shock waves, heavy lines, in the physical plane. Each of the following three figures gives the variation of the physical quantities, S, r^+, p, c on either particle paths or C^+ characteristics. Thus the solution to the piston problem is furnished entirely in graphical form.

$M_0=3$

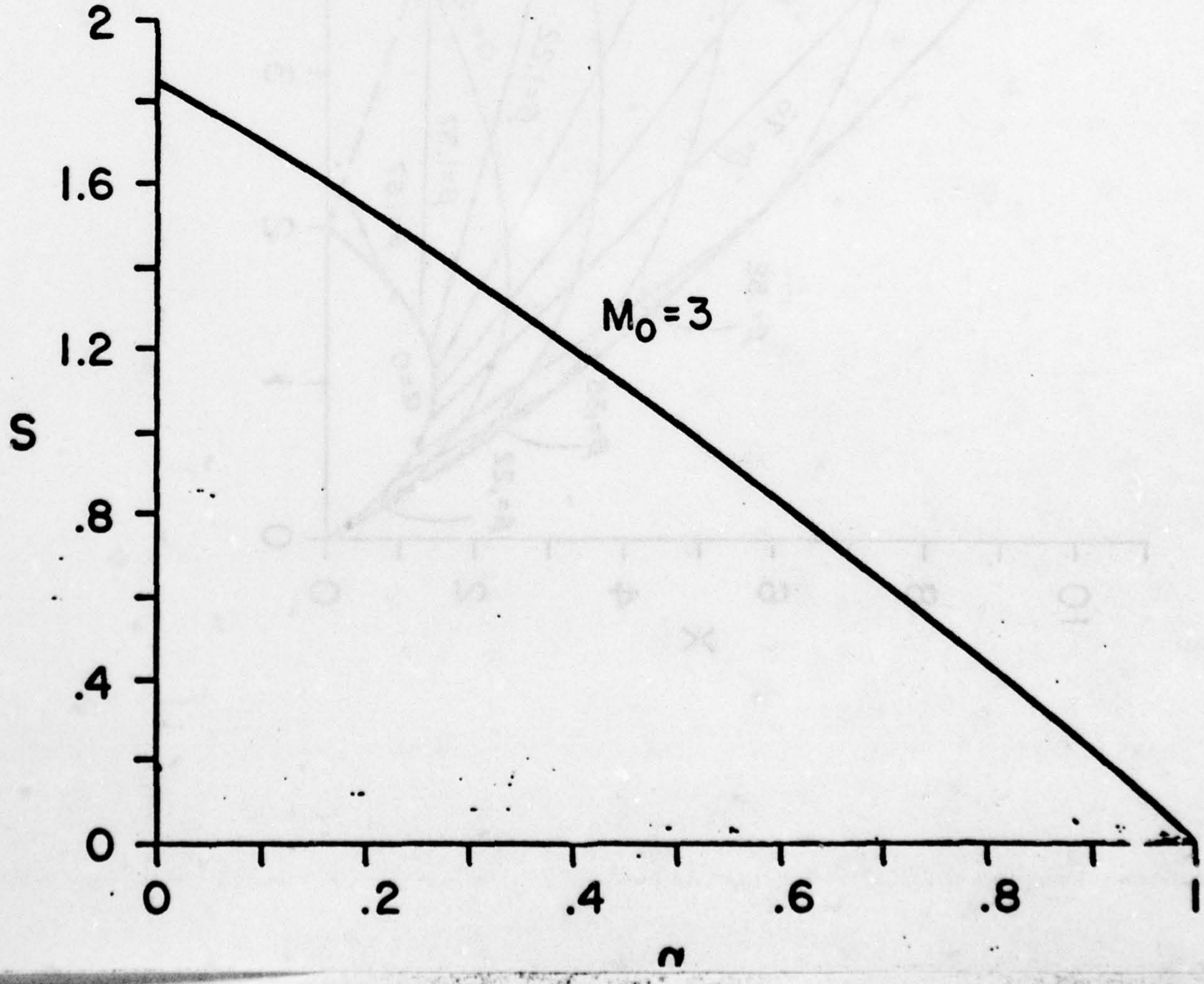


p
u
α
β=.22
β=.35
β=.52
β=.76
β=.91
β=1.07
β=1.22
β=1.37
β=1.67

$M_0=3$



r⁻
β
α
α=.8
α=.6
α=.4
α=.2
α=0



S
α
M₀=3

The results describe in this section are reported on, in detail, in the following two manuscripts:

L. Sirovich and T.H. Chong, An Approximate Solution in Gasdynamics.

T.H. Chong and L. Sirovich, On the Numerical Integration of the Gasdynamic Equations.

Both of these manuscripts have been submitted for publication.

References

[1] W.D. Hayes and R. Probstein, Hypersonic Flow Theory (2nd. Ed.), Academic Press, New York, (1966), p. 497ff.

[2] J.J. Mahony, J. Aeronat. Aci. 22, 673-680 (1955).

[3] R. Meyer, Quart. Appl. Math. 14, 433-436 (1957).

3. Steady Supersonic Flows

In parallel with the treatment described in Sec. 2, we have also investigated the case of two-dimensional supersonic flows. The analysis and methods summarized below applied to flows ranging from the low supersonic limit to high supersonic speeds.

The two-dimensional form of the gasdynamic equations in characteristic form follows from the introduction of the Prandtl angle

$$\mu = \sin^{-1}\frac{1}{M}$$

and the flow angle

$$\theta = \tan^{-1}\frac{u}{v}$$

as dependent variables. Then with the introduction of the Riemann invariants

(1) $$r^{\pm} = \theta \pm P(u)$$

where the Prandtl function P is defined by

(2) $$P = \frac{\gamma+1}{\gamma-1}\tan^{-1}\left(\frac{\gamma+1}{\gamma-1}\tan\mu\right) - \mu ,$$

the governing equations are:

(3) $$\frac{\partial S}{\partial \beta} = 0$$

(4) $$r^{+}_{\alpha} = \frac{1}{\gamma}\sin\mu\cos\mu\,\frac{dS}{d\alpha}$$

(5) $$r^{-}_{\beta} = (1 - \tan\theta\tan\mu)\,\frac{x_{\beta}}{x_{\alpha}}\,\theta_{\alpha}$$

In these the physical plane (x, y) has been transformed to (α, β)-plane by means of

(6) $y_\beta = \tan \theta \, x_\beta$

(7) $y_\alpha = \tan(\theta + \mu) x_\alpha$

The first of these corresponds to streamlines and the second to C^+ characteristics.

The method of solution (in the upper half plane) follows from what we refer to as the approximat ion.of shock expansion theory [1-3], viz. that $\bar{r}$ is "almost constant" on streamlines. We therefore replace Eq. (6) with the condition

(8) $\bar{r} = \bar{r}(\alpha)$

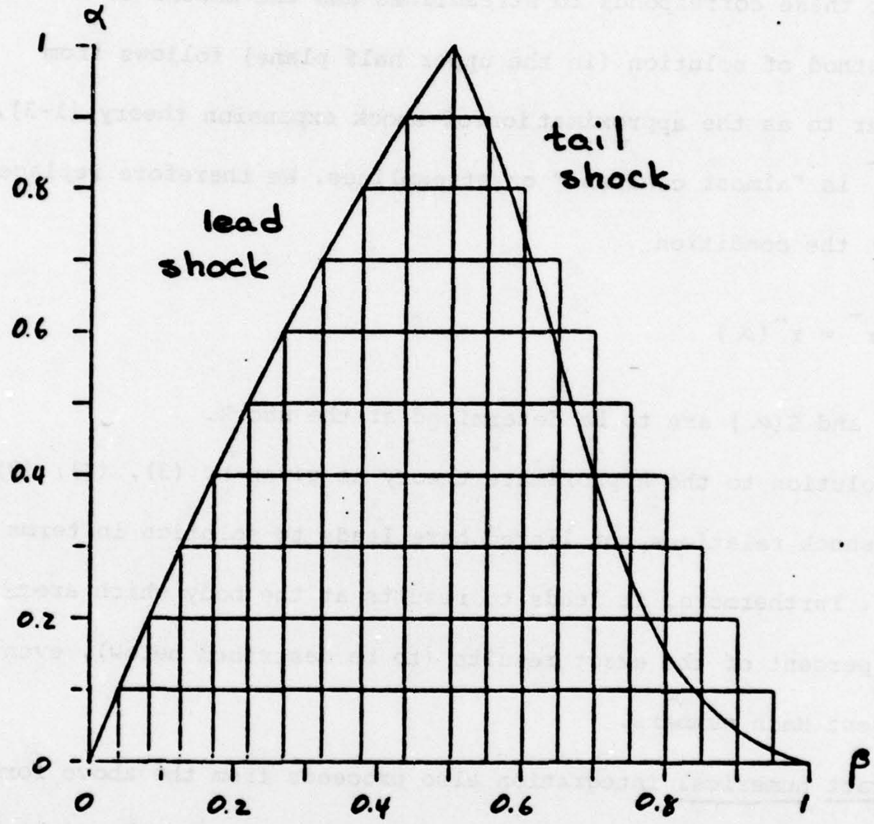
Both $\bar{r}(\alpha)$ and $S(\alpha)$ are to be determined at the shock.

The solution to the approximate theory as given by (3), (4), (8), as well as shock relations not listed here leads to solution in terms of quadratures. Furthermore, it leads to results at the body which are within one percent of the exact results (to be described below), even at the highest Mach numbers.

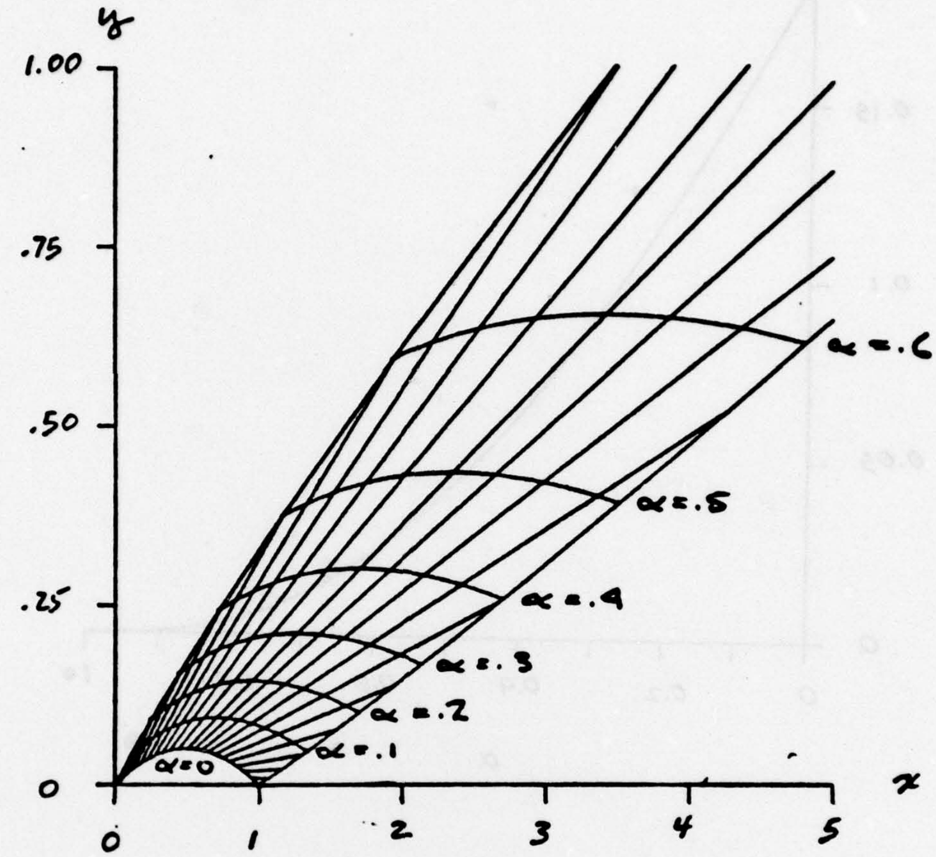
An <u>exact</u> <u>numerical</u> integration also proceeds from the above formulation. The results of the approximate theory are substituted into the right hand side of Eq. (5) which leads to a new determination of $\bar{r}(\alpha, \beta)$ (note that $\bar{r}$ is now β dependent). This in turn is used to solve the remaining equations (5), (6), (7), and (3), as well as the shock relations.

A significant aspect of the numerical integration is the fact that the shock wave is fixed as part of the transformation from the

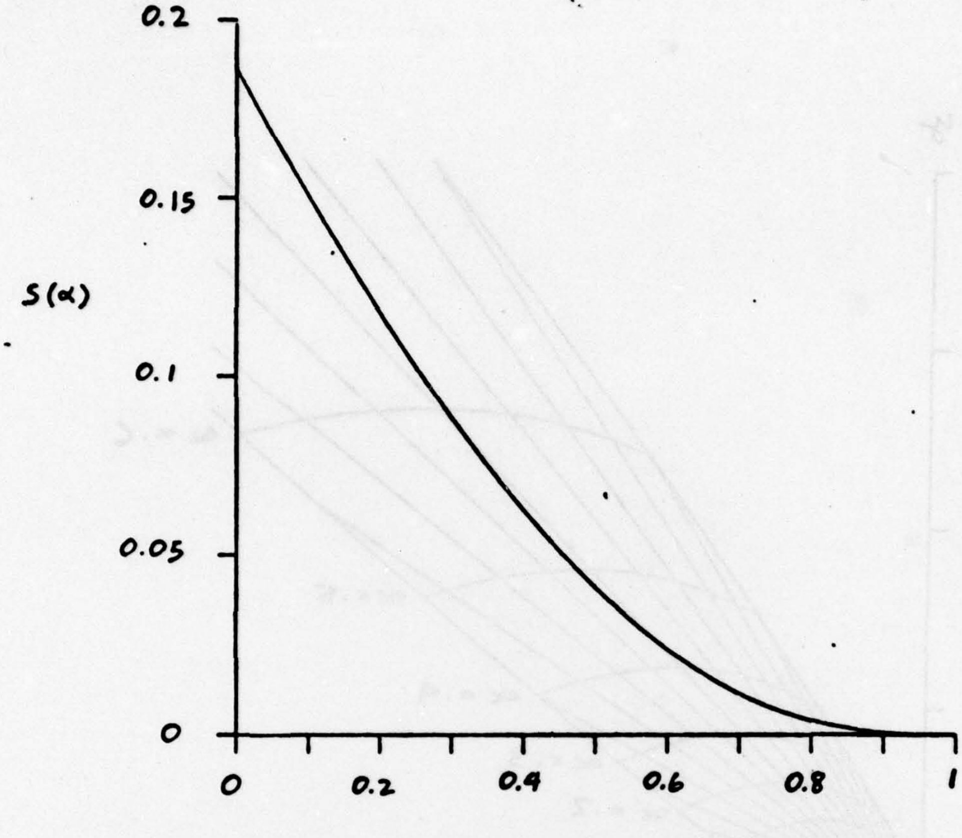
physical plane (x, y) to the (α, β) plane. The figure below indicates the numerical grid system. The lead shock is fixed and a uniform α grid is chosen. The remaining grid system is then fixed. (The grid shown is roughly 1/10 the density used.)

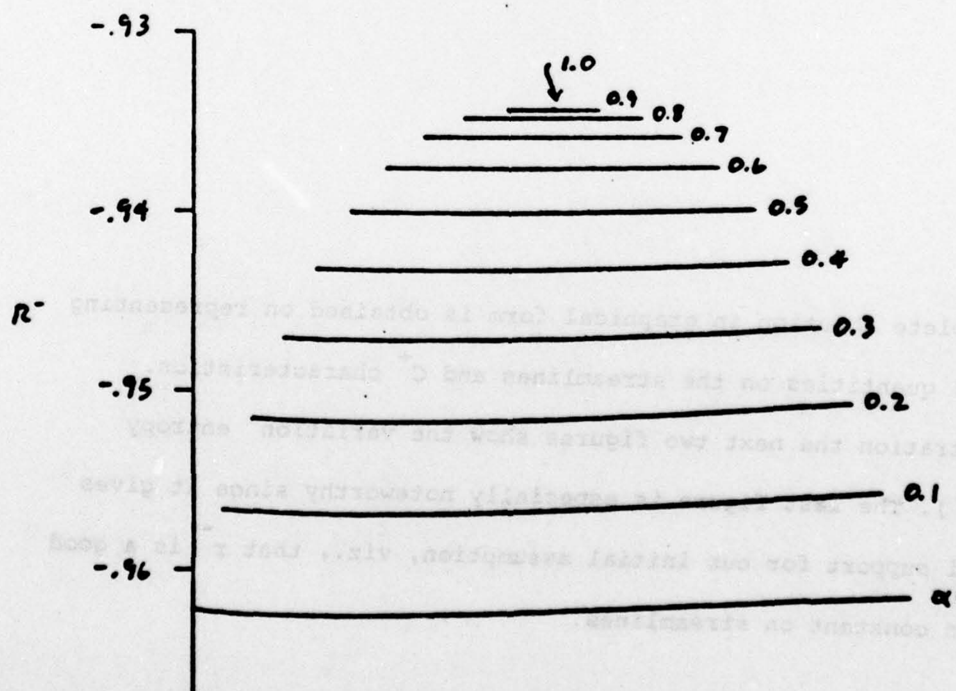


To illustrate these results we consider a 10% parabolic profile and an upstream Mach number of five. The following figure shows the map of the previous figure to the physical plane. α equal constant are streamlines (not all of which are drawn), and β equal constant are C^+ characteristics.



A complete solution in graphical form is obtained on representing the physical quantities on the streamlines and C^+ characteristics. As an illustration the next two figures show the variation entropy and $r^-(\alpha, \beta)$. The last figure is especially noteworthy since it gives a posteriori support for our initial assumption, viz., that r^- is a good approxiation constant on streamlines.


0.2
0.15
S(α)
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
α


-.93
-.94
-.95
-.96
-.97
R⁻
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
α=0.0
0
0.2
0.4
0.6
0.8
1
β

A number of additional cases have been worked out and further study of these cases should be made.

A report on the progress made so far is now in preparation.

References

[1] W.D. Hayes, and R. Probstein, Hypersonic Flow Theory (2nd ed.), Academic Press, New York, (1966) p. 497ff.

[2] J.J. Mahony, J. Aeronaut. Sci. 22, 673-680 (1955).

[3] R. Meyer, Quart. Appl. Math. 14, 433-436 (1957).

4. Aerodynamic Wakes

We have considered the development of a wake which follows the passage of a lifting body. As is well known, a persistent vortical structure known as "horseshoe vortices" is found in such a wake. Our efforts have been directed at determining simplified equations which govern this wake structure. We have, however, included compressibility, entropy, and dissipative effects in the calculations.

As a starting point of the analysis, it is assumed that the streamwise direction, x, experiences slow variation as compared with variation in the cross (Treffitz) plane, (y, z). The method of treating this problem is based on two scale perturbation techniques. To summarize the results we denote the crosswise velocity by $u_{\perp} = (v, w)$ and the longitudinal velocity by u . It is then found that

(1) $$\nabla_{\perp} \cdot u_{\perp} = 0$$

where $\nabla_{\perp} = (\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ so that a crosswise stream function ψ exists, i.e.

$$(u, w) = (-\frac{\partial}{\partial z}\psi, \frac{\partial}{\partial y}\psi)$$

The stream function is then found to satisfy

(2) $$[\frac{\partial}{\partial x} + (u_{\perp} \cdot \nabla_{\perp})]\,\omega = \mathcal{J}\nabla_{\perp}^{2}$$

where $\omega = -\nabla_{\perp}^{2}$ is the Treffitz plane vorticity and $\mathcal{J}$ is a viscous coefficient. The longitudinal velocity u also satisfies an equation of the same form,

$$(3) \qquad \left[\frac{\partial}{\partial x} + (u_\perp \cdot \nabla_\perp)\right]u = \mathcal{J}\, \nabla_\perp^2 u$$

In contrast to (2) and (3) the temperature is found to be governed by

$$(4) \qquad \left[\frac{\partial}{\partial x} + (u_\perp \cdot \nabla_\perp)\right]T = \xi\, \nabla_\perp^2 T$$

where ξ is a heat conductivity coefficient.

As equations (2) and (3) indicate, vorticity and entropy effects decouple and give rise to different diffusion rates. The total vorticity to the present order is given by

$$\left(\omega, \frac{-\partial u}{\partial z}, \frac{\partial u}{\partial y}\right)$$

and as a result the twisting and stretching of vortex lines occurs within the framework of this analysis.

The work just described is independent of Mach number and is thus equally valid for supersonic as well as subsonic flows.

In addition to the above work we have examined the effect of a real atmosphere on the forms of the governing equation. Statification effects and shearing winds have been included in a further analysis. The resulting equations are again in the form of nonlinear diffusion equations--but are now completely coupled.

A full report of this work appears in the manuscript:

L. Sirovich and T.H. Chong, On the Equations Governing a Trailing Wake which has been submitted for publication.

5. Two-Dimensional Unsteady Flow

Consider uniform (upstream) two-dimensional inviscid flow past a body which is undergoing unsteady motion. Assume that the motion of the body is "slow" in the sense that the reduced frequency

$$\frac{\omega L}{U}$$

is small. ω is a typical oscillation frequency of the body motion, L is the body length, and U is the supersonic uniform upstream flow. From the assumed weakness of the shock waves, we also assume that the Riemann invariant along the upstream going characteristic takes a negligible jump across the front shock (see Sec. 2). Then from simple wave transonic perturbation theory, we obtain the governing equations for the normalized velocity perturbations, (u, v), in the form

(1)
$$v + \frac{2}{3\Gamma}(\beta^2+\Gamma u)^{3/2} = 2\beta^3/3\Gamma$$

$$v_t + F_x + G_y = 0$$

where

$$F = \frac{2\beta^5}{5M^2\Gamma}\left\{1-\left(1-\frac{3\Gamma v}{2\beta^3}\right)^{5/3}\right\}$$

$$G = \frac{\beta^4}{2M^2\Gamma}\left[1-\left(1-\frac{3\Gamma v}{2\beta^3}\right)^{4/3}\right]$$

and $M = U/c_o$ is the upstream unperturbed Mach number, $\beta^2 = M^2-1$, $\Gamma = (\gamma+1)M^2$.

The boundary condition on v is

(2)
$$v = \sigma f_\sigma(\sigma,\tau) + f_\tau(\sigma,\tau)$$

where $y = \mathcal{F}(x, t)$ is the body and

$$\begin{aligned}\sigma &= x - y F'(v)/G'(v) \\ \tau &= t - y/G'(v)\end{aligned} \tag{3}$$

From shock analysis, we find that the shock surface is governed by an equation of the form

$$A(y,\sigma,\tau)\, y_\sigma + B(y,\sigma,\tau)\, y_\tau = C(y,\sigma,\tau) \tag{4}$$

where the functions A, B, and C are explicit. Eqns. (3) and (4) together with the solution for v, (2), define a shock surface in the physical plane.

For purposes of illustration, we further assume that v is small (this puts the analysis in the supersonic regime), and expand F and G in powers of v, then (4) yields (up to quadratic terms in v) a solution in the form

$$\begin{aligned}\sigma &= -\beta^2/s M^2 + \tau_0(\zeta) \\ \tau &= s + \tau_0(\zeta) \\ y &= \left[\exp\left(\int b\,ds\right)\right] \int_0^s \left[\exp\left[-\int b\,ds\right]\right] a\,ds + y_0(\zeta)\end{aligned} \tag{5}$$

where s is arc length, $(\sigma_0(\zeta), \tau_0(\zeta), y_0(\zeta))$ parameterizes a curve on the shock surface, and the functions a and b are explicit. By eliminating parameters s and ζ from (5), we obtain an approximate solution to (4) in the form $y = y(\sigma, \tau)$.

As a sample calculation, we have considered a pulsating wing defined by

$$y = \epsilon x(1-x)(1+\delta \sin \omega t)$$

for $0 \leq \epsilon, 0 \leq x \leq 1, 0 \leq \delta < 1$. In this case all steps outlined above can be carried out explicitly to yield a shock surface as well as conditions on the moving body.

At the present stage of development we do not feel that this work warrants publication.

In a similar manner two-dimensional supersonic theory was considered. From this analysis we found that Whitham theory (1st order) with only minor modification may be extended to second order. This theory still places characteristics and shock waves incorrectly to this order. A new equation which has the form (in two dimensions)

$$\alpha(\varphi, M)\varphi_y + \beta(\varphi, M)\varphi_x = 0$$

has been derived and shown to be valid uniformly through the second order. This equation may be integrated by standard means.

Although this work may be of analytic interest, it is superseded by the work described in Sec. 3. As a result we have not pursued it further and do not contemplate a manuscript on this topic.

Personnel

During the course of this contract the following scientific staff was employed: L. Sirovich, T.H. Chong, and T. Lewis. A part-time office assistant (C. Mencke) was also employed. In addition, various freelance help was employed for preparation of text and figures for manuscripts.